ON A LIMITING RELATION BETWEEN RAMANUJAN'S ENTIRE FUNCTION $A_q(z)$ AND θ-FUNCTION

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Abstract. We will use a discrete analogue of the classical Laplace method to show that the main term of the asymptotic expansions of certain entire functions, including Ramanujan's entire function $A_q(z)$, can be expressed in terms of θ-functions.

1. Introduction

Throughout the paper, we assume that

$$0 < q < 1. \tag{1}$$

For any complex number a, we define [5, 8, 12]

$$(a; q)_\infty = \prod_{k=0}^{\infty}(1 - aq^k) \tag{2}$$

and the q-shifted factorial as

$$(a; q)_n = \frac{(a; q)_\infty}{(aq^n; q)_\infty} \tag{3}$$

for any integer n. Assume that $|z| < 1$, the q-Binomial theorem is [5, 8, 12]

$$\frac{(az; q)_\infty}{(z; q)_\infty} = \sum_{k=0}^{\infty} \frac{(a; q)_k}{(q; q)_k} z^k, \tag{4}$$

which defines an analytic function in the region $|z| < 1$. Its limiting case

$$(z; q)_\infty = \sum_{k=0}^{\infty} \frac{q^{k(k-1)/2}}{(q; q)_k}(-z)^k \tag{5}$$

is one of many q-exponential identities. Ramanujan's entire function $A_q(z)$ is defined as [12]

$$A_q(z) = \sum_{k=0}^{\infty} \frac{q^{k^2}}{(q; q)_k}(-z)^k. \tag{6}$$

It is known that $A_q(z)$ has infinitely many positive zeros and satisfies the following three term recurrence

$$A_q(z) - A_q(qz) + qzA_q(q^2z) = 0. \tag{7}$$

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Ramanujan function $A_q(z)$, which is also called q-Airy function in the literature, appears repeatedly in Ramanujan's work starting from the Rogers-Ramanujan identities, where $A_q(-1)$ and $A_q(-q)$ are expressed as infinite products, [2], to properties of and conjectures about its zeros, [4, 5, 9, 15]. It is called q-Airy function because it appears repeatedly in the Plancherel-Rotach type asymptotics [11, 16, 17] of q-orthogonal polynomials, just like classical Airy function in the classical Plancherel-Rotach asymptotics of classical orthogonal polynomials [23, 12]. However, Ramanujan's $A_q(z)$ is not the q-analogue of classical Airy functions. Since

$$\frac{1-q^k}{1-q} \geq kq^{k-1}, \tag{8}$$

for $k = 1, 2, ...$, then

$$\left| \frac{(1-q)^k}{(q;q)_k} q^{k^2} (-z)^k \right| \leq \frac{(q\,|z|)^k}{k!} \tag{9}$$

for $k = 0, 1, \ldots$, for any complex number z, applying Lebesgue's dominated convergent theorem we have

$$\lim_{q\to 1} A_q((1-q)z) = e^{-z}. \tag{10}$$

We also have obtained the inequality

$$|A_q((1-q)z)| \leq e^{q|z|} \tag{11}$$

for any complex number z. For any nonzero complex number z, we define the theta function as

$$\theta(z;q) = \sum_{k=-\infty}^{\infty} q^{k^2/2} z^k. \tag{12}$$

Jacobi's triple product formula says that [5, 8, 12]

$$\sum_{k=-\infty}^{\infty} q^{k^2/2} z^k = (q, -q^{1/2}z, -q^{1/2}/z; q)_\infty. \tag{13}$$

For any real number x, then,

$$x = \lfloor x \rfloor + \{x\}, \tag{14}$$

where the fractional part of x is $\{x\} \in [0, 1)$ and $\lfloor x \rfloor \in \mathbb{Z}$ is the greatest integer less or equal x. The arithmetic function

$$\chi(n) = 2\left\{\frac{n}{2}\right\} = n - 2\left\lfloor \frac{n}{2} \right\rfloor, \tag{15}$$

which is the principal character modulo 2,

$$\chi(n) = \begin{cases} 1 & 2 \nmid n \\ 0 & 2 \mid n \end{cases}. \tag{16}$$

For any positive real number t, we consider the following set

$$\mathbb{S}(t) = \{\{nt\} : n \in \mathbb{N}\}. \tag{17}$$

It is clear that $\mathbb{S}(t) \subset [0, 1)$ and it is a finite set when t is a positive rational number. In this case, for any $\lambda \in \mathbb{S}(t)$, there are infinitely many positive integers n and m such that

$$nt = m + \lambda, \tag{18}$$

where

$$m = \lfloor nt \rfloor . \tag{19}$$

If t is a positive irrational number, then $\mathbb{S}(t)$ is a subset of $(0,1)$ with infinite elements, and it is well-known that $\mathbb{S}(t)$ is uniformly distributed in $(0,1)$. A theorem of Chebyshev [10] says that given any $\beta \in [0,1)$, there are infinitely many positive integers n and m such that

$$nt = m + \beta + \gamma_n \tag{20}$$

with

$$|\gamma_n| \le \frac{3}{n}. \tag{21}$$

For n large enough, this implies

$$m = \lfloor nt \rfloor . \tag{22}$$

We will also make use of the trivial inequalities

$$|e^x - 1| \le |x| e^{|x|} \tag{23}$$

for any $x \in \mathbb{C}$, and

$$e^{-x} \ge 1 - x \tag{24}$$

for $0 < x < 1$. The following lemma is from [17].

Lemma 1.1. *Given any $n \in \mathbb{N}$, if $a > 0$,*

$$\frac{(a;q)_\infty}{(a;q)_n} = (aq^n;q)_\infty = 1 + R_1(a;n) \tag{25}$$

with

$$|R_1(a;n)| \le \frac{(-aq^2;q)_\infty aq^n}{1-q}. \tag{26}$$

While for $0 < aq < 1$,

$$\frac{(a;q)_n}{(a;q)_\infty} = \frac{1}{(aq^n;q)_\infty} = 1 + R_2(a;n) \tag{27}$$

with

$$|R_2(a;n)| \le \frac{aq^n}{(1-q)(aq;q)_\infty}. \tag{28}$$

Proof. It is clear from (5) that

$$R_1(a;n) = \sum_{k=1}^{\infty} \frac{q^{k(k-1)/2}(-aq^n)^k}{(q;q)_k} = -aq^n \sum_{k=0}^{\infty} \frac{q^{k(k+1)/2}(-aq^n)^k}{(q;q)_{k+1}}.$$

Hence for $a > 0$, we have

$$\begin{aligned} \left| \sum_{k=0}^{\infty} \frac{q^{k(k+1)/2}(-aq^n)^k}{(q;q)_{k+1}} \right| &\le \frac{1}{1-q} \sum_{k=0}^{\infty} \frac{q^{k(k-1)/2}}{(q;q)_k} (aq^{n+1})^k \\ &\le \frac{(-aq^2;q)_\infty}{1-q}, \end{aligned}$$

and (26) follows. Moreover from (4)

$$R_2(a;n) \quad = \quad aq^n \sum_{k=0}^{\infty} \frac{(aq^n)^k}{(q;q)_{k+1}},$$

hence

$$|R_2(a;n)| \quad \le \quad \frac{aq^n}{(1-q)(aq^n;q)_\infty} \le \frac{aq^n}{(1-q)(aq;q)_\infty},$$

which is (28) and the proof of the lemma is complete. □

2. Ramanujan's Entire Function $A_q(z)$

For Ramanujan's entire function $A_q(z)$, we have the following:

Theorem 2.1. *Given an arbitrary nonzero complex number u, we have*

(1) *For any positive rational number t and $\lambda \in \mathbb{S}(t)$, there are infinitely many positive integers n and m such that*

$$tn = m + \lambda. \tag{29}$$

For each such λ, n and m we have

$$A_q(q^{-nt}u) = \frac{(-u)^{\lfloor m/2 \rfloor}\left\{\theta(-u^{-1}q^{\chi(m)+\lambda};q^2) + r(n)\right\}}{(q;q)_\infty q^{\lfloor m/2 \rfloor (nt - \lfloor m/2 \rfloor)}} \tag{30}$$

with

$$|r(n)| \le \frac{3(-q^3;q)_\infty \theta(|u|^{-1};q)}{1-q} \times \left\{ q^{m/4} + \frac{q^{m^2/16}}{|u|^{\lfloor m/4 \rfloor + 1}} \right\}. \tag{31}$$

for n sufficiently large.

(2) *For any positive irrational number t and $\beta \in [0,1)$, there are infinitely many positive integers n and m such that*

$$nt = m + \beta + \gamma_n \tag{32}$$

with

$$|\gamma_n| \le \frac{3}{n}. \tag{33}$$

Let

$$\nu_n = \left\lfloor -\frac{q^2 \log n}{\log q} \right\rfloor \gg 4, \tag{34}$$

for each such β, n and m we have

$$A_q(q^{-nt}u) = \frac{(-u)^{\lfloor m/2 \rfloor}\left\{\theta(-u^{-1}q^{\chi(m)+\beta};q^2) + e(n)\right\}}{(q;q)_\infty q^{\lfloor m/2 \rfloor (nt - \lfloor m/2 \rfloor)}}. \tag{35}$$

and

$$|e(n)| \le \frac{48(-q^3;q)_\infty \theta(|u|^{-1};q)}{(1-q)} \times \left\{ \frac{\log n}{n} + q^{\nu_n^2/2}|u|^{\nu_n} + \frac{q^{\nu_n^2/2}}{|u|^{1+\nu_n}} \right\}. \tag{36}$$

for n sufficiently large.

Proof. From (6), we have

$$A_q(q^{-nt}u) = \sum_{k=0}^{\infty} \frac{q^{k^2-knt}}{(q;q)_k}(-u)^k. \tag{37}$$

The classical Laplace method is used to study the asymptotics for

$$\int_{-\infty}^{\infty} e^{\lambda f(x)} dx, \tag{38}$$

as $\lambda \to +\infty$, for example, see [24]. If the real function $f(x)$ has some maximas, from the nature of $e^{f(x)}$, when λ is large, then the major contributions of the integral come from the neighbourhood of these maximas. We break the integral into several pieces so that each piece has only one maxima and then replace the integrands by simpler functions within each subintegrals to get the asymptotics formula. Our situation is very similar here. We notice that q^{k^2-knt} has maximum around $\frac{nt}{2}$, just as in the Laplace method for (38), we break the sum into two subsums and estimate them respectively.

In the case that t is a positive rational number, for any $\lambda \in \mathbb{S}(t)$ and n, m are large, we have

$$\begin{aligned} A_q(q^{-nt}u)(q;q)_\infty &= \sum_{k=0}^{\infty}(q^{k+1};q)_\infty q^{k^2-km-k\lambda}(-u)^k \\ &= s_1 + s_2 \end{aligned} \tag{39}$$

where

$$s_1 = \sum_{k=0}^{\lfloor m/2 \rfloor} (q^{k+1};q)_\infty q^{k^2-km-k\lambda}(-u)^k \tag{40}$$

and

$$s_2 = \sum_{k=\lfloor m/2 \rfloor+1}^{\infty} (q^{k+1};q)_\infty q^{k^2-km-k\lambda}(-u)^k. \tag{41}$$

In s_1 we reverse the order of summation to obtain

$$\begin{aligned}
\frac{s_1 q^{\lfloor m/2\rfloor(nt-\lfloor m/2\rfloor)}}{(-u)^{\lfloor m/2\rfloor}} &= \sum_{k=0}^{\lfloor m/2\rfloor}(q^{\lfloor m/2\rfloor-k+1};q)_\infty q^{k^2}(-u^{-1}q^{\chi(m)+\lambda})^k \\
&= \sum_{k=0}^{\infty} q^{k^2}(-u^{-1}q^{\chi(m)+\lambda})^k \\
&- \sum_{k=\lfloor m/4\rfloor+1}^{\infty} q^{k^2}(-u^{-1}q^{\chi(m)+\lambda})^k \\
&+ \sum_{k=0}^{\lfloor m/4\rfloor} q^{k^2}(-u^{-1}q^{\chi(m)+\lambda})^k \left((q^{\lfloor m/2\rfloor-k+1};q)_\infty - 1\right) \\
&+ \sum_{k=\lfloor m/4\rfloor+1}^{\lfloor m/2\rfloor}(q^{\lfloor m/2\rfloor-k+1};q)_\infty q^{k^2}(-u^{-1}q^{\chi(m)+\lambda})^k \\
&= \sum_{k=0}^{\infty} q^{k^2}(-u^{-1}q^{\chi(m)+\lambda})^k + s_{11}+s_{12}+s_{13}. \qquad (42)
\end{aligned}$$

Since

$$0 < (q^{\lfloor m/2\rfloor-k+1};q)_\infty < 1 \qquad (43)$$

for $0 \le k \le \lfloor m/2\rfloor$, then,

$$\begin{aligned}
|s_{11}+s_{13}| &\le 2\sum_{k=\lfloor m/4\rfloor+1}^{\infty} q^{k^2}|u|^{-k} \\
&\le 2\sum_{k=\lfloor m/4\rfloor+1}^{\infty} q^{k^2/2}|u|^{-k} \\
&\le \frac{2q^{m^2/16}}{|u|^{\lfloor m/4\rfloor+1}}\theta(|u|^{-1};q). \qquad (44)
\end{aligned}$$

By (26), for $0 \le k \le \lfloor m/4\rfloor$, we have

$$\left|(q^{\lfloor m/2\rfloor-k+1};q)_\infty - 1\right| \le \frac{(-q^3;q)_\infty}{1-q}q^{m/4}, \qquad (45)$$

then

$$\begin{aligned}
|s_{12}| &\le \frac{(-q^3;q)_\infty q^{m/4}}{1-q}\sum_{k=0}^{\infty} q^{k^2}|u|^{-k} \\
&\le \frac{(-q^3;q)_\infty q^{m/4}}{1-q}\sum_{k=0}^{\infty} q^{k^2/2}|u|^{-k} \\
&\le \frac{q^{m/4}(-q^3;q)_\infty\theta(|u|^{-1};q)}{1-q}, \qquad (46)
\end{aligned}$$

hence

$$\frac{s_1 q^{\lfloor m/2\rfloor(nt-\lfloor m/2\rfloor)}}{(-u)^{\lfloor m/2\rfloor}} = \sum_{k=0}^{\infty} q^{k^2}(-u^{-1}q^{\chi(m)+\lambda})^k + r_1(n) \qquad (47)$$

with

$$|r_1(n)| \le \frac{2(-q^3;q)_\infty \theta(|u|^{-1};q)}{1-q} \times \left\{ q^{m/4} + \frac{q^{m^2/16}}{|u|^{\lfloor m/4 \rfloor + 1}} \right\}. \tag{48}$$

In s_2 we change the summation from k to $k + \lfloor m/2 \rfloor$

$$\begin{aligned} \frac{s_2 q^{\lfloor m/2 \rfloor (nt - \lfloor m/2 \rfloor)}}{(-u)^{\lfloor m/2 \rfloor}} &= \sum_{k=1}^{\infty} (q^{\lfloor m/2 \rfloor + k + 1}; q)_\infty q^{k^2} (-uq^{-\chi(m)-\lambda})^k \\ &= \sum_{k=1}^{\infty} q^{k^2} (-uq^{-\chi(m)-\lambda})^k \\ &+ \sum_{k=1}^{\infty} q^{k^2} (-uq^{-\chi(m)-\lambda})^k \left[(q^{\lfloor m/2 \rfloor + k + 1}; q)_\infty - 1 \right] \\ &= \sum_{k=-\infty}^{-1} q^{k^2} (-u^{-1} q^{\chi(m)+\lambda})^k + r_2(n). \end{aligned} \tag{49}$$

By (26), for $k \ge 1$

$$\left| (q^{\lfloor m/2 \rfloor + k + 1}; q)_\infty - 1 \right| \le \frac{q^{\lfloor m/2 \rfloor + k + 1} (-q^3;q)_\infty}{1-q} \le \frac{(-q^3;q)_\infty q^{m/2+k}}{1-q}, \tag{50}$$

then

$$\begin{aligned} |r_2(n)| &\le \frac{(-q^3;q)_\infty q^{m/2}}{1-q} \sum_{k=1}^{\infty} q^{k^2} (|u| q^{1-\chi(m)-\lambda})^k \\ &\le \frac{(-q^3;q)_\infty q^{m/2}}{1-q} \sum_{k=1}^{\infty} q^{k^2/2 + k^2/2 - k} |u|^k \\ &\le \frac{(-q^3;q)_\infty q^{m/2-1/2}}{1-q} \sum_{k=1}^{\infty} q^{k^2/2} |u|^k \\ &= \frac{(-q^3;q)_\infty q^{m/2-1/2}}{1-q} \sum_{k=-1}^{-\infty} q^{k^2/2} |u|^{-k} \\ &\le \frac{q^{m/4} (-q^3;q)_\infty \theta(|u|^{-1};q)}{1-q}. \end{aligned} \tag{51}$$

Thus we have proved that

$$A_q(q^{-nt}u) = \frac{(-u)^{\lfloor m/2 \rfloor} \left\{ \theta(-u^{-1} q^{\chi(m)+\lambda}; q^2) + r(n) \right\}}{(q;q)_\infty q^{\lfloor m/2 \rfloor (nt - \lfloor m/2 \rfloor)}} \tag{52}$$

with

$$|r(n)| \le \frac{3(-q^3;q)_\infty \theta(|u|^{-1};q)}{1-q} \times \left\{ q^{m/4} + \frac{q^{m^2/16}}{|u|^{\lfloor m/4 \rfloor + 1}} \right\}. \tag{53}$$

for n, m and λ satisfying (29) with n and m are sufficiently large.

In the case that t is a positive irrational number, for any real number $\beta \in [0,1)$, when n and m are sufficiently large and satisfy (32) and (33) with

$$1 > \beta + \gamma_n > -1, \tag{54}$$

then

$$2 > \chi(m) + \beta + \gamma_n > -1. \tag{55}$$

For these integers n, we take

$$\nu_n = \left\lfloor -\frac{q^2 \log n}{\log q} \right\rfloor \gg 4. \tag{56}$$

Then,

$$\begin{aligned} A_q(q^{-nt}u)(q;q)_\infty &= \sum_{k=0}^{\infty} (q^{k+1};q)_\infty q^{k^2-km-k\beta-k\gamma_n}(-u)^k \\ &= s_1 + s_2, \end{aligned} \tag{57}$$

with

$$s_1 = \sum_{k=0}^{\lfloor m/2 \rfloor} (q^{k+1};q)_\infty q^{k^2-km-k\beta-k\gamma_n}(-u)^k \tag{58}$$

and

$$s_2 = \sum_{k=\lfloor m/2 \rfloor+1}^{\infty} (q^{k+1};q)_\infty q^{k^2-km-k\beta-k\gamma_n}(-u)^k. \tag{59}$$

In s_1 we reverse the order of summation to get,

$$\begin{aligned} \frac{s_1 q^{\lfloor m/2 \rfloor (nt - \lfloor m/2 \rfloor)}}{(-u)^{\lfloor m/2 \rfloor}} &= \sum_{k=0}^{\lfloor m/2 \rfloor} (q^{\lfloor m/2 \rfloor - k+1};q)_\infty q^{k^2} (-u^{-1} q^{\chi(m)+\beta+\gamma_n})^k \\ &= \sum_{k=0}^{\infty} q^{k^2} (-u^{-1} q^{\chi(m)+\beta})^k \\ &- \sum_{k=\nu_n+1}^{\infty} q^{k^2} (-u^{-1} q^{\chi(m)+\beta})^k \\ &+ \sum_{k=0}^{\nu_n} q^{k^2} (-u^{-1} q^{\chi(m)+\beta})^k \left(q^{k\gamma_n} - 1\right) \\ &+ \sum_{k=0}^{\nu_n} q^{k^2} (-u^{-1} q^{\chi(m)+\beta+\gamma_n})^k \left\{ (q^{\lfloor m/2 \rfloor - k+1};q)_\infty - 1 \right\} \\ &+ \sum_{k=\nu_n+1}^{\lfloor m/2 \rfloor} q^{k^2} (-u^{-1} q^{\chi(m)+\beta+\gamma_n})^k (q^{\lfloor m/2 \rfloor - k+1};q)_\infty \\ &= \sum_{k=0}^{\infty} q^{k^2} (-u^{-1} q^{\chi(m)+\beta})^k + s_{11} + s_{12} + s_{13} + s_{14}. \end{aligned} \tag{60}$$

Since

$$0 < (q^{\lfloor m/2 \rfloor - k+1};q)_\infty < 1 \tag{61}$$

for $\nu_n + 1 \le k \le \lfloor m/2 \rfloor$ and

$$0 < q^{\chi(m)+\beta+\gamma_n} \le q^{-1}, \tag{62}$$

then,

$$(\nu_n + 1)^2 - (\nu_n + 1) > \frac{(\nu_n + 1)^2}{2}, \tag{63}$$

then,

$$\begin{aligned}
|s_{11} + s_{14}| &\le \sum_{k=\nu_n+1}^{\infty} q^{k^2} |u|^{-k} + \sum_{k=\nu_n+1}^{\infty} q^{k^2} |u|^{-k} q^{-k} \\
&\le \frac{2q^{(\nu_n+1)^2-\nu_n-1}}{|u|^{\nu_n+1}} \sum_{k=0}^{\infty} q^{k^2} (|u|^{-1} q^{2\nu_n+1})^k \\
&\le \frac{2q^{(\nu_n+1)^2-\nu_n-1}}{|u|^{\nu_n+1}} \sum_{k=0}^{\infty} q^{k^2/2} |u|^{-k} \\
&\le \frac{2q^{\nu_n^2/2}}{|u|^{1+\nu_n}} \theta(|u|^{-1}; q)
\end{aligned} \tag{64}$$

Since $0 < q < 1$ and $\lim_{n\to\infty} \log n/n = 0$, there exists a positive integer N such that for $n \ge N$

$$n \ge \frac{3q^2 \log n}{\log q^{-1}}. \tag{65}$$

Hence by (23) and (33)

$$|q^{k\gamma_n} - 1| \le \nu_n |\gamma_n| e^{\nu_n |\gamma_n|} \le \frac{3q^2 e}{\log q^{-1}} \frac{\log n}{n}. \tag{66}$$

for $n \ge N$ and $0 \le k \le \nu_n$. Thus

$$\begin{aligned}
|s_{12}| &\le \frac{3eq^2}{\log q^{-1}} \frac{\log n}{n} \sum_{k=0}^{\infty} q^{k^2} |u|^{-k} \\
&\le \frac{3e}{\log q^{-1}} \frac{\log n}{n} \sum_{k=0}^{\infty} q^{k^2/2} |u|^{-k} \\
&\le \frac{12\theta(|u|^{-1}; q)}{\log q^{-1}} \frac{\log n}{n}
\end{aligned} \tag{67}$$

From (26),

$$\begin{aligned}
|(q^{\lfloor m/2 \rfloor - k+1}; q)_\infty - 1| q^{-k} &\le \frac{(-q^3; q)_\infty q^{\lfloor m/2 \rfloor - 2k+1}}{1-q} \\
&\le \frac{(-q^3; q)_\infty q^{m/4}}{1-q}
\end{aligned} \tag{68}$$

for $0 \le k \le \nu_n$ and n sufficiently large. Thus

$$\begin{aligned} |s_{13}| &\le \frac{(-q^3;q)_\infty q^{m/4}}{1-q} \sum_{k=0}^{\nu_n} q^{k^2} |u|^{-k} \\ &\le \frac{(-q^3;q)_\infty q^{m/4}}{1-q} \sum_{k=0}^{\nu_n} q^{k^2/2} |u|^{-k} \\ &\le \frac{(-q^3;q)_\infty \theta(|u|^{-1};q)}{1-q} q^{m/4} \\ &\le \frac{(-q^3;q)_\infty \theta(|u|^{-1};q)}{1-q} \frac{\log n}{n}, \end{aligned} \tag{69}$$

for n sufficiently large.

$$\frac{s_1 q^{\lfloor m/2 \rfloor (nt - \lfloor m/2 \rfloor)}}{(-u)^{\lfloor m/2 \rfloor}} = \sum_{k=0}^{\infty} q^{k^2} (-u^{-1} q^{\chi(m)+\beta})^k + e_1(n) \tag{70}$$

with

$$\begin{aligned} |e_1(n)| &\le \frac{24(-q^3;q)_\infty \theta(|u|^{-1};q)}{1-q} \\ &\times \left\{ \frac{1-q+\log q^{-1}}{2\log q^{-1}} \frac{\log n}{n} + \frac{q^{\nu_n^2/2}}{|u|^{1+\nu_n}} \right\} \\ &\le \frac{24(-q^3;q)_\infty \theta(|u|^{-1};q)}{1-q} \\ &\times \left\{ \frac{\log n}{n} + \frac{q^{\nu_n^2/2}}{|u|^{1+\nu_n}} \right\}. \end{aligned} \tag{71}$$

In (71), we have used the inequality (24) for $x = 1-q$ to show that $1-q < \log q^{-1}$.

Similarly, in s_2 we change summation from k to $k + \lfloor m/2 \rfloor$,

$$\begin{aligned} \frac{s_2 q^{\lfloor m/2 \rfloor (nt - \lfloor m/2 \rfloor)}}{(-u)^{\lfloor m/2 \rfloor}} &= \sum_{k=1}^{\infty} q^{k^2} (-uq^{-\chi(m)-\beta})^k \\ &- \sum_{k=\nu_n+1}^{\infty} q^{k^2} (-uq^{-\chi(m)-\beta})^k \\ &+ \sum_{k=1}^{\nu_n} q^{k^2} (-uq^{-\chi(m)-\beta})^k (q^{-k\gamma_n} - 1) \\ &+ \sum_{k=1}^{\infty} q^{k^2} (-uq^{1-\chi(m)-\beta-\gamma_n})^k q^{-k} \left\{ (q^{\lfloor m/2 \rfloor + k + 1};q)_\infty - 1 \right\} \\ &+ \sum_{k=\nu_n+1}^{\infty} q^{k^2} (-uq^{-\chi(m)-\beta-\gamma_n})^k \\ &= \sum_{k=-\infty}^{-1} q^{k^2} (-u^{-1} q^{\chi(m)+\beta})^k + s_{21} + s_{22} + s_{23} + s_{24}. \end{aligned} \tag{72}$$

Just as we has done with the sum s_1, we can show that

$$
\begin{aligned}
|s_{21}+s_{24}| &\le 2\sum_{k=\nu_n+1}^{\infty} q^{k^2-2k}|u|^k\\
&\le 2q^{\nu_n^2/2}|u|^{\nu_n}\sum_{k=1}^{\infty} q^{k^2+2\nu_n k}|u|^k\\
&\le 2q^{\nu_n^2/2}|u|^{\nu_n}\sum_{k=1}^{\infty} q^{k^2/2}|u|^k\\
&\le 2q^{\nu_n^2/2}|u|^{\nu_n}\sum_{k=-1}^{-\infty} q^{k^2/2}|u|^{-k}\\
&\le 2q^{\nu_n^2/2}|u|^{\nu_n}\theta(|u|^{-1};q), \qquad (73)
\end{aligned}
$$

$$
\begin{aligned}
|s_{22}| &\le \sum_{k=1}^{\nu_n} q^{k^2/2+k^2/2-2k}|u|^k|q^{-k\gamma_n}-1|\\
&\le \frac{3q^2e}{\log q^{-1}}\frac{\log n}{n}q^{-2}\sum_{k=1}^{\nu_n} q^{k^2/2}|u|^k\\
&\le \frac{3e}{\log q^{-1}}\frac{\log n}{n}\sum_{k=-1}^{-\infty} q^{k^2/2}|u|^{-k}\\
&\le \frac{12\theta(|u|^{-1};q)}{\log q^{-1}}\frac{\log n}{n}, \qquad (74)
\end{aligned}
$$

$$
\begin{aligned}
|s_{23}| &\le \frac{(-q^3;q)_\infty q^{m/2}}{1-q}\sum_{k=1}^{\infty} q^{k^2/2+k^2/2-k}|u|^k\\
&\le \frac{(-q^3;q)_\infty q^{m/2-1/2}}{1-q}\sum_{k=-1}^{-\infty} q^{k^2/2}|u|^{-k}\\
&\le \frac{(-q^3;q)_\infty \theta(|u|^{-1};q)}{1-q}\frac{\log n}{n} \qquad (75)
\end{aligned}
$$

for n sufficiently large. Thus

$$
\frac{s_2 q^{\lfloor m/2\rfloor(nt-\lfloor m/2\rfloor)}}{(-u)^{\lfloor m/2\rfloor}} = \sum_{k=-\infty}^{-1} q^{k^2}(-u^{-1}q^{\chi(m)+\beta})^k + e_2(n) \qquad (76)
$$

with

$$
\begin{aligned}
|e_2(n)| &\le \frac{24(-q^3;q)_\infty\theta(|u|^{-1};q)}{(1-q)}\\
&\quad\times\left\{\frac{\log n}{n}+q^{\nu_n^2/2}|u|^{\nu_n}\right\}. \qquad (77)
\end{aligned}
$$

Thus we have

$$
A_q(q^{-nt}u) = \frac{(-u)^{\lfloor m/2\rfloor}\left\{\theta(-u^{-1}q^{\chi(m)+\beta};q^2)+e(n)\right\}}{(q;q)_\infty q^{\lfloor m/2\rfloor(nt-\lfloor m/2\rfloor)}} \qquad (78)
$$

with

$$e(n) = e_1(n) + e_2(n) \tag{79}$$

and

$$\begin{aligned} |e(n)| \le & \frac{48(-q^3;q)_\infty \theta(|u|^{-1};q)}{(1-q)} \\ & \times \left\{ \frac{\log n}{n} + q^{\nu_n^2/2}|u|^{\nu_n} + \frac{q^{\nu_n^2/2}}{|u|^{1+\nu_n}} \right\}. \end{aligned} \tag{80}$$

for n sufficiently large. □

Remark 2.2. We have the following remarks on Theorem 2.1:

(1) (7), (30) and (31) imply the trivial formula for $\theta(z;q)$

$$\theta(z;q) = zq^{1/2}\theta(zq;q). \tag{81}$$

(2) We can rewrite (30) and (31) into the form

$$\theta(-u^{-1}q^{\chi(m)+\lambda};q^2) = \frac{A_q(q^{-nt}u)(q;q)_\infty q^{\lfloor m/2\rfloor(nt-\lfloor m/2\rfloor)}}{(-u)^{\lfloor m/2\rfloor}} + o(1),$$

where the $o(1)$ is uniform for q in any compact subset of $(0,1)$ and u in any compact subset of $\mathbb{C}\setminus\{0\}$.

(3) We can rewrite the formulas (35) and (36) into the form

$$\theta(-u^{-1}q^{\chi(m)+\beta};q^2) = \frac{(q;q)_\infty A_q(q^{-nt}u)q^{\lfloor m/2\rfloor(nt-\lfloor m/2\rfloor)}}{(-u)^{\lfloor m/2\rfloor}} + o(1),$$

where the $o(1)$ is uniform for q in any compact subset of $(0,1)$ and u in any compact subset of $\mathbb{C}\setminus\{0\}$.

3. A Class of Entire Functions

The phenomenon demonstrated with $A_q(z)$ in Theorem 2.1 is universal for a class of entire basic hypergeometric function of type

$$f(z) = \sum_{k=0}^\infty \frac{(a_1,\dots,a_r;q)_k q^{lk^2}}{(b_1,\dots,b_s;q)_k} z^k, \tag{82}$$

with $l>0$, where

$$(a_1,\dots,a_r;q)_k = \prod_{j=1}^r (a_j;q)_k. \tag{83}$$

A confluent basic hypergeometric series is formally defined as

$$ {}_m\phi_n\left(\begin{array}{c} a_1,\dots,a_m \\ b_1,\dots,b_n \end{array} \middle| \, q, z \right) = \sum_{k=0}^\infty \frac{(a_1,\dots,a_m;q)_k}{(q,b_1,\dots,b_n;q)_k} z^k \left(-q^{(k-1)/2}\right)^{k(m+1-n)}, \tag{84}$$

when $m+1-n>0$. It is clear that the function

$$ {}_m\phi_n\left(\begin{array}{c} a_1,\dots,a_m \\ b_1,\dots,b_n \end{array} \middle| \, q, -zq^{(m+1-n)/2} \right) \tag{85}$$

is of the form (82) with

$$l = \frac{m+1-n}{2}. \tag{86}$$

For a clean statement of the following theorem, we also define

$$c(r,s;q) := \frac{(b_1,\dots,b_s;q)_\infty}{(a_1,\dots,a_r;q)_\infty}. \tag{87}$$

Theorem 3.1. *Assume that*

$$0 \le a_1,\dots,a_r,b_1,\dots,b_s < 1. \tag{88}$$

and

$$u \in \mathbb{C}\setminus\{0\}. \tag{89}$$

We have

(1) *For any positive rational number t and $\lambda \in \mathbb{S}(t)$, there are infinitely many positive integers n and m such that*

$$tn = m + \lambda. \tag{90}$$

For each such λ, n and m we have

$$f(q^{-lnt}u) = \frac{u^{\lfloor m/2\rfloor}\left\{\theta(u^{-1}q^{l\chi(m)+l\lambda};q^{2l}) + r(n)\right\}}{c(r,s;q)q^{l\lfloor m/2\rfloor(nt-\lfloor m/2\rfloor)}} \tag{91}$$

with

$$\begin{aligned}|r(n)| \quad &\le \quad \left(\frac{2}{1-q}\right)^{r+s+1}\frac{\prod_{j=1}^{s}(-b_jq^2;q)_\infty\theta(|u|^{-1};q^l)}{\prod_{j=1}^{r}(a_j;q)_\infty} \\ &\times \quad \left\{q^{m/4} + \frac{q^{lm^2/16}}{|u|^{\lfloor m/4\rfloor+1}}\right\}.\end{aligned} \tag{92}$$

for n sufficiently large.

(2) *For any positive irrational number t and $\beta \in [0,1)$, there are infinitely many positive integers n and m such that*

$$nt = m + \beta + \gamma_n \tag{93}$$

with

$$|\gamma_n| \le \frac{3}{n}. \tag{94}$$

Let

$$\nu_n = \left\lfloor -\frac{q^2\log n}{\log q}\right\rfloor \gg 4, \tag{95}$$

for each such β, n and m we have

$$f(q^{-lnt}u) = \frac{u^{\lfloor m/2\rfloor}\left\{\theta(u^{-1}q^{l\chi(m)+l\beta};q^{2l}) + e(n)\right\}}{c(r,s;q)q^{l\lfloor m/2\rfloor(nt-\lfloor m/2\rfloor)}}, \tag{96}$$

with

$$\begin{aligned}|e(n)| &\le \frac{48\prod_{j=1}^{s}(-b_jq^2;q)_\infty\theta(|u|^{-1};q^l)}{(1-q)\prod_{j=1}^{r}(a_j;q)_\infty} \\ &\times\left\{\frac{\log n}{n} + q^{l\nu_n^2/2}|u|^{\nu_n} + \frac{q^{l\nu_n^2/2}}{|u|^{1+\nu_n}}\right\}.\end{aligned} \tag{97}$$

for n sufficiently large.

Proof. The function

$$f(q^{-lnt}u)c(r,s;q) = \sum_{k=0}^{\infty}\prod_{j=1}^{s}\{1+R_1(b_j;k)\}\prod_{j=1}^{r}\{1+R_2(a_j;k)\}\, q^{l(k^2-ntk)}u^k$$

$$= s_1 + s_2, \tag{98}$$

where

$$s_1 = \sum_{k=0}^{\lfloor m/2\rfloor}\prod_{j=1}^{s}\{1+R_1(b_j;k)\}\prod_{j=1}^{r}\{1+R_2(a_j;k)\}\, q^{l(k^2-ntk)}u^k \tag{99}$$

and

$$s_2 = \sum_{k=\lfloor m/2\rfloor+1}^{\infty}\prod_{j=1}^{s}\{1+R_1(b_j;k)\}\prod_{j=1}^{r}\{1+R_2(a_j;k)\}\, q^{l(k^2-ntk)}u^k. \tag{100}$$

By Lemma 1.1, we have

$$\left|\prod_{j=1}^{s}\{1+R_1(b_j;k)\}\prod_{j=1}^{r}\{1+R_2(a_j;k)\}-1\right| \le \left(\frac{2}{1-q}\right)^{r+s}\frac{\prod_{j=1}^{s}(-b_jq^2;q)_\infty q^k}{\prod_{j=1}^{r}(a_j;q)_\infty}, \tag{101}$$

and

$$\left|\prod_{j=1}^{s}\{1+R_1(b_j;k)\}\prod_{j=1}^{r}\{1+R_2(a_j;k)\}\right| \le \frac{1}{\prod_{j=1}^{r}(a_j;q)_\infty}. \tag{102}$$

The rest of the proof is very similar to the proof for Theorem 2.1. □

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